$U(\mathrm{so}(8,\mathbb{C}))$向量表示的范畴化

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摘 要：为了范畴化 $U(\mathrm{so}(8,\mathbb{C}))$向量表示的 n 次张量积，定义了一般线性李代数 $\mathfrak{gl}_n$ 的伯恩斯坦-盖尔芬德-盖尔芬德（Bernstein-Gelfand-Gelfand，BGG）范畴 $\mathscr{O}$ 的若干子范畴，这些子范畴 Grothendieck 群的复化范畴化了 D_4 型李代数包络代数向量表示 n 次张量积的底空间；定义了 BGG 范畴 $\mathscr{O}$ 上的一系列投射函子用于范畴化 $U(\mathrm{so}(8,\mathbb{C}))$在张量积上的作用；得到 $h_i(1\leqslant i\leqslant 4)$可由一对函子$(\mathscr{H}_i^+,\mathscr{H}_i^-)(1\leqslant i\leqslant 4)$范畴化，$e_i,f_i(1\leqslant i\leqslant 3)$分别由 ε_i、$\mathscr{F}_i(1\leqslant i\leqslant 3)$范畴化，$e_4,f_4$ 分别由一对函子$(\varepsilon_4^+,\varepsilon_4^-)(\mathscr{F}_4^+,\mathscr{F}_4^-)$范畴化.

关键词：向量表示；范畴化；BGG 范畴；投射函子
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Categorification of the Vector Representation of $U(\mathrm{so}(8,\mathbb{C}))$

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Abstract: To categorify the n-tensor products of vector representation of $U(\mathrm{so}(8,\mathbb{C}))$, some subcategoriees of Bernstein-Gelfand-Gelfand (BGG) category $\mathscr{O}$ of the general linear Lie algebra $\mathfrak{gl}_n$ were defined. The complexifications of their Grothendieck groups were used for categorifing base spaces of n-tensor products of vector representation. And some projective endfunctors of BGG category $\mathscr{O}$, which were used for categorifing the action of $U(\mathrm{so}(8,\mathbb{C}))$ on n-tensor products of vector representation, were defined. It was got that $h_i(1\leqslant i\leqslant 4)$ can be categorified by a pair of functors $(\mathscr{H}_i^+,\mathscr{H}_i^-)(1\leqslant i\leqslant 4)$, $e_i,f_i(1\leqslant i\leqslant 3)$, can be categorified by $\varepsilon_i,\mathscr{F}_i(1\leqslant i\leqslant 3)$, e_4,f_4 can be categorified by a pair of functors $(\varepsilon_4^+,\varepsilon_4^-)$, $(\mathscr{F}_4^+,\mathscr{F}_4^-)$, respectively.

Key words: vector representation; categorification; BGG category; projective functors

范畴化的思想最早是由 L. Crane 等[1]引出的，其基本思想是用较复杂的对象研究简单对象. 而李代数的 BGG 范畴是范畴化常用的来源. M. Khovanov 等[2]给出利用 BGG 范畴范畴化对称群 S_n 表示的许多例子. M. Khovanov 等[3]又利用 BGG 范畴范畴化了对称群的 Specht 表示. J. N. Berstein 等[4]利用 BGG 范畴范畴化了 $U(\mathrm{sl}2)$基本表示的 n 次张量积. Xu 等[5]利用 BGG 范畴范畴化了 $U(\mathrm{so}(7,\mathbb{C}))$旋表示的 n 次张量积. 本文利用 BGG 范畴范畴化 $U(\mathrm{so}(8,\mathbb{C}))$向量表示的 n 次张量积.

1 预备知识

本文所有的向量空间和代数都定义在复数域$\mathbb{C}$上. 用 $K(\mathscr{A})$表示 Abelian 范畴 $\mathscr{A}$ 的 Grothendieck 群. 对任意的 $M\in\mathscr{A}$，用$[M]$表示 M 在 $K(\mathscr{A})$中的像. 对任意的正合函子 F，用$[M]$表示 Abelian 群间的群同态. 设$\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$是有限维约化李代数$\mathfrak{g}$关于固定 Cartan 子代数 $\mathfrak{h}$的三角分解. $U(\mathfrak{g})$表示$\mathfrak{g}$

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的包络代数，$Z(\mathscr{g})$表示 $U(\mathscr{g})$的中心. Θ 表示所有中心特征构成的集合. W 表示 $\mathscr{g}$ 的 Weyl 群，ρ 表示所有正根和的一半. 对任意的 $w\in W,\lambda\in\mathscr{h}^*$，$W$ 在 $\mathscr{h}^*$ 上的点作用为 $w\lambda=w(\lambda+\rho)-\rho$. $M(\lambda)$表示最高权为 $\lambda\in\mathscr{h}^*$ 的 Verma 模. θ_λ 表示对应 λ 的中心特征. 设 Λ^+ 表示所有支配权构成的集合(关于点作用). 根据文献[6]可知 Λ^+ 和 Θ 间存在双射 η. 设 $\mathscr{gl}_n=n^+\oplus\mathscr{h}\oplus n^-$ 是 $\mathscr{gl}_n$ 的三角分解，$\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 为 $\mathbb{R}^n$ 中的标准正交基. 将 $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^n$ 与 $\mathscr{h}^*$ 等同，这样 $\mathscr{gl}_n$ 的正根集为 $\{\varepsilon_i-\varepsilon_j|1\leqslant i<j\leqslant n\}$，并且其单根集为 $\{\varepsilon_i-\varepsilon_{i+1}|1\leqslant i\leqslant n-1\}$. $\mathscr{gl}_n$ 的 Weyl 群同构于对称群 S_n，其中生成元 s_i 置换 ε_i、ε_{i+1}. 设 L_n 是 $\mathscr{gl}_n$ 的 n-维基本表示，$u_1,u_2,\cdots,u_n$ 分别是对应权 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 的权向量. 易知 L_n^* 的权为 $-\varepsilon_1,-\varepsilon_2,\cdots,-\varepsilon_n$. 由 $\{u_i\otimes u_i|1\leqslant i\leqslant n\}$ 和 $\{u_i\otimes u_j+u_j\otimes u_i|1\leqslant i<j\leqslant n\}$ 生成 $L_n\otimes L_n$ 的子表示为 L_n 的对称方 Sym^2L_n，由 $\{u_i\otimes u_j-u_j\otimes u_i|1\leqslant i<j\leqslant n\}$ 生成 $L_n\otimes L_n$ 的子表示为 L_n 的交错方 Alt^2L_n. 类似可定义 L_n^* 的对称方 $\mathrm{Sym}^2L_n^*$ 和交错方 $\mathrm{Alt}^2L_n^*$. 易得 Sym^2L_n 的权为 $\{2\varepsilon_i|1\leqslant i\leqslant n\}$ 和 $\{\varepsilon_i+\varepsilon_j|1\leqslant i<j\leqslant n\}$，$\mathrm{Alt}^2L_n$ 权为 $\{\varepsilon_i+\varepsilon_j|1\leqslant i<j\leqslant n\}$. 对任意的 $U(\mathscr{g})$-模 V，为方便令 $F_V:=V\otimes-$. 用 $\mathscr{O}$ 表示 $\mathscr{gl}_n$ 的 BGG 范畴. F 是 $\mathscr{O}$ 的自函子，如果存在有限维 $U(\mathscr{g})$-模 V 使得 F 同构于 F_V 的直和项，称 F 为投射函子[7].

对 $M\in ob\mathscr{O},\theta\in\Theta$，定义 M 的子模 $M_\theta=\{v\in M|$ 对任意的 $z\in Z(\mathscr{g})$ 存在 $n\in\mathbb{Z}$ 使得 $(z-\theta(z))^nv=0\}$. 设 $\mathscr{O}_\theta$ 为所有 M_θ 组成 $\mathscr{O}$ 的全子范畴. 由文献[7]知 $\mathscr{O}=\bigoplus_{\theta=\theta_\lambda}\mathscr{O}_\theta$. 对 $M=\bigoplus_{\theta=\theta_\lambda}M_\theta$，存在 $\mathscr{O}$ 的自投射函子 Proj_θ 定义为 $\mathrm{Proj}_\theta(M)=M_\theta$.

命题 1[6]　关于投射函子下面结论成立：

1) 投射函子是正合的；

2) 投射函子的直和仍是投射函子；

3) 投射函子的合成仍是投射函子；

4) 设 F、G 是 2 个投射函子，如果$[F]=[G]$，则 F 和 G 是自然同构的.

任取 $\theta\in\Theta$，由文献[9]知 $K(\mathscr{O}_\theta)$ 的 $\mathbb{Z}$-基为 $\{[M(\lambda)]|\theta=\theta_\lambda\}=\{[M(\mu)]|\mu\in W\lambda\}$，$K(\mathscr{O})$的 $\mathbb{Z}$ 基为 $\{[M(\lambda)]|\exists\theta\in\Theta\text{s. t. }\theta=\theta_\lambda\}$.

命题 2[4]　设 V 是有限维 $U(\mathscr{g})$-模，$v_1,v_2,\cdots,v_n$ 分别是对应权 $\mu_1,\mu_2,\cdots,\mu_n$ 的权向量，则有$[V\otimes M(\lambda)]=\sum_{i=1}^{n}[M(\lambda+\mu_i)]$.

定义 1[8]　特殊正交李代数 $\mathrm{so}(8,\mathbb{C})$ 的包络代数 $U(\mathrm{so}(8,\mathbb{C}))$ 定义为由 e_i、f_i、$h_i(1\leqslant i\leqslant4)$ 生成满足以下条件的结合代数：

1) $h_ih_j=h_jh_i,1\leqslant i,j\leqslant4$；

2) $e_if_j-f_je_i=\delta_{ij}h_i,1\leqslant i,j\leqslant4$；

3) $h_ie_j-e_jh_i=a_{ij}e_i,1\leqslant i,j\leqslant4$；

4) $h_if_j-f_jh_i=-a_{ij}f_i,1\leqslant i,j\leqslant4$；

5) $\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}e_i^{1-a_{ij}-k}e_je_i^k=0,i\neq j$;

6) $\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}f_i^{1-a_{ij}-k}f_jf_i^k=0,i\neq j$.

其中 a_{ij} 属于 $\mathrm{so}(8,\mathbb{C})$ 的 Cartan 矩阵 A.

定义 2[8]　设 $V=\bigoplus_{i=1}^{8}\mathbb{C}v_i$ 是复数域 $\mathbb{C}$ 上的 8-维向量空间，定义

$$e_1v_i=v_{i+1},i=1,7;\ e_2v_i=v_{i+1},i=2,6$$
$$e_3v_i=v_{i+1},i=3,5;\ e_4v_i=v_{i+2},i=3,4$$
$$f_1v_i=v_{i-1},i=2,8L;\ f_2v_i=v_{i-1},i=3,7$$
$$f_3v_i=v_{i-1},i=4,6;\ f_4v_i=v_{i-2},i=5,6$$

其他 e_i、f_j 的作用为零；

$$h_iv_j=(e_if_i-f_ie_i)v_j,1\leqslant i\leqslant4,1\leqslant j\leqslant8$$

按上述作用 V 作成 $U(\mathrm{so}(8,\mathbb{C}))$ 的表示，称其为向量表示.

2　向量表示的范畴化

在本节给出向量表示的范畴化. 首先设

$$A=\{\bar{a}=(a_1,\cdots,a_n)|1\leqslant a_i\leqslant8,\ 1\leqslant i\leqslant n\}$$

$$D=\left\{\bar{d}=(d_1,\cdots,d_8)|d_i\in\mathbb{Z}\geqslant0,1\leqslant i\leqslant8,\ \sum_{i=1}^{8}d_i=n\right\}$$

任取 $\bar{d}$、$\bar{d}'\in D$，定义等价关系 $\sim$，$\bar{d}\sim\bar{d}'\Leftrightarrow$

$$\begin{cases}-d_1+d_2-d_7+d_8=-d_1'+d_2'-d_7'+d_8'\\-d_2+d_3-d_6+d_7=-d_2'+d_3'-d_6'+d_7'\\-d_3+d_4+d_5+d_6=-d_3'+d_4'+d_5'+d_6'\\-d_3-d_4+d_5+d_6=-d_3'-d_4'+d_5'+d_6'\end{cases}$$

用$[\bar{d}]$表示 $\bar{d}$ 所在的等价类，$\widetilde{D}$ 表示所有等价类的集合. 任取 $\bar{d}=(d_1,d_2,\cdots,d_8)\in D$，令

$$\bar{d}_i=(d_1,\cdots,d_{i-1},d_i-1,d_{i+1},\cdots,d_8)$$
$$\bar{d}^i=(d_1,\cdots,d_{i-1},d_i+1,d_{i+1},\cdots,d_8)$$

任取 $\bar{a}=(a_1,a_2,\cdots,a_n)\in A$，定义

$$d_k^{\bar{a}}=\#\{a_m|a_m=k,1\leqslant m\leqslant n\}$$

易知 $\bar{d}_{\bar{a}}=(d_1^{\bar{a}},d_2^{\bar{a}},\cdots,d_8^{\bar{a}})\in D$. 令

$B_{[\bar{d}]}:=\{v_{a_1}\otimes\cdots\otimes v_{a_n}|\bar{a}=(a_1,a_2,\cdots,a_n)\in A,$

$\bar{d}_{\bar{a}} \in [\bar{d}]\}$

设$^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$是由 $B_{[\bar{d}]}$ 张成的$\mathbb{Z}$-模. 令

$$^{\mathbb{Z}}(V^{\otimes n}) = \bigoplus_{[\bar{d}] \in \tilde{D}} {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$$

$$(V^{\otimes n})_{[\bar{d}]} = \mathbb{C} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]},\ V^{\otimes n} = \mathbb{C} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}(V^{\otimes n})$$

易知 $V^{\otimes n}$具有权空间分解 $V^{\otimes n} = \bigoplus_{[\bar{d}] \in \tilde{D}} (V^{\otimes n})_{[\bar{d}]}$. 任取 $\bar{a} = (a_1, a_2, \cdots, a_n) \in A, \bar{d} \in D$,令

$$M(a_1, a_2, \cdots, a_n) := M(a_1 \varepsilon_1 + \cdots + a_n \varepsilon_n - \rho)$$

$$\lambda_{\bar{d}} = \sum_{i=0}^{7} (8 - i) \sum_{j=1}^{d_{8-i}} \varepsilon_{d_8 + d_7 + \cdots + d_{9-i} + j}$$

且令 $\theta_{\bar{d}}$表示 $\lambda_{\bar{d}} - \rho$ 在 η 下的像. 下面定义 BGG 范畴 $\mathscr{O}$的若干子范畴:

$$\mathscr{O}_{\bar{d}} := \mathscr{O}_{\theta_{\bar{d}}},\ \mathscr{O}_{[\bar{d}]} := \bigoplus_{\bar{d}' \in [\bar{d}]} \mathscr{O}_{\bar{d}'},\ \mathscr{O}^n := \bigoplus_{[\bar{d}] \in \tilde{D}} \mathscr{O}_{[\bar{d}]}$$

作为线性空间向量表示 V 的范畴化可由下面的定理给出.

定理 1[5] 任取 $\bar{a} = (a_1, a_2, \cdots, a_n) \in A$,定义

$$\gamma_n([M(a_1, a_2, \cdots, a_n)]) = v_{a_1} \otimes \cdots \otimes v_{a_n}$$

则 γ_n 是 $K(\mathscr{O}^n)$到$^{\mathbb{Z}}(V^{\otimes n})$的阿贝尔群同构,并且将 γ_n 限制到 $K(\mathscr{O}_{[\bar{d}]})$上得到 $K(\mathscr{O}_{[\bar{d}]})$,$^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$间的阿贝尔群同构.

任取 $\bar{d} = (d_1, d_2, \cdots, d_8) \in D$,定义

$$c_1(\bar{d}) = d_8 - d_7 + d_2 - d_1$$

$$c_2(\bar{d}) = d_7 - d_6 + d_3 - d_2$$

$$c_3(\bar{d}) = d_6 + d_5 + d_4 - d_3$$

$$c_4(\bar{d}) = d_6 + d_5 - d_4 - d_3$$

令

$$\mathscr{H}^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}]) := (\mathrm{Id}_{\mathscr{O}_{[\bar{d}]}})^{\oplus c_i(\bar{d}) \mathrm{sgn}(c_i(\bar{d}))}: \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\bar{d}]}$$

其中 $\mathrm{Id}_{\mathscr{O}_{[\bar{d}]}}$是 $\mathscr{O}_{[\bar{d}]}$上的恒等函子,$\mathrm{sgn}(c_i(\bar{d}))$($i = 1, 2, 3, 4$)是 $c_i(\bar{d})$的符号函数. 根据等价关系 ~ 的定义,易知 $\mathscr{H}^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}])$的定义与$[\bar{d}]$中代表元的选择无关.

对任意的 $\bar{d} = (d_1, d_2, \cdots, d_8) \in D$,定义下面的正合函子(正合性由命题 1 可知).

$$\varepsilon_1^1(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_1^2}} \circ F_{L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_1^2}$$

$$\varepsilon_1^7(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_7^8}} \circ F_{L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_7^8}$$

$$\varepsilon_2^2(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_2^3}} \circ F_{L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_2^3}$$

$$\varepsilon_2^6(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_6^7}} \circ F_{L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_6^7}$$

$$\varepsilon_3^3(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_3^4}} \circ F_{L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_3^4}$$

$$\varepsilon_3^5(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_5^6}} \circ F_{L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_5^6}$$

$$\varepsilon_4^{+3}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_3^5}} \circ F_{\mathrm{sym}^2 L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_3^5}$$

$$\varepsilon_4^{+4}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_4^6}} \circ F_{\mathrm{sym}^2 L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_4^6}$$

$$\varepsilon_4^{-3}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_3^5}} \circ F_{\mathrm{alt}^2 L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_3^5}$$

$$\varepsilon_4^{-4}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_4^6}} \circ F_{\mathrm{alt}^2 L_n}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_4^6}$$

根据等价关系 ~ 的定义,令

$$[\overleftarrow{d}_1] := [\bar{d}_1^2] = [\bar{d}_7^8],\ [\overleftarrow{d}_2] := [\bar{d}_2^3] = [\bar{d}_6^7]$$

$$[\overleftarrow{d}_3] := [\bar{d}_3^4] = [\bar{d}_5^6],\ [\overleftarrow{d}_4] := [\bar{d}_4^6] = [\bar{d}_3^5]$$

定义

$$\varepsilon_1([\bar{d}]) := \bigoplus_{\bar{d}' \in [\bar{d}]} (\varepsilon_1^1(\bar{d}') \oplus \varepsilon_1^7(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overleftarrow{d}_1]}$$

$$\varepsilon_2([\bar{d}]) := \bigoplus_{\bar{d}' \in [\bar{d}]} (\varepsilon_2^2(\bar{d}') \oplus \varepsilon_2^6(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overleftarrow{d}_2]}$$

$$\varepsilon_3([\bar{d}]) := \bigoplus_{\bar{d}' \in [\bar{d}]} (\varepsilon_3^3(\bar{d}') \oplus \varepsilon_3^5(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overleftarrow{d}_3]}$$

$$\varepsilon_4^+([\bar{d}]) := \bigoplus_{\bar{d}' \in [\bar{d}]} (\varepsilon_4^{+3}(\bar{d}') \oplus \varepsilon_4^{+4}(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overleftarrow{d}_4]}$$

$$\varepsilon_4^-([\bar{d}]) := \bigoplus_{\bar{d}' \in [\bar{d}]} (\varepsilon_4^{-3}(\bar{d}') \oplus \varepsilon_4^{-4}(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overleftarrow{d}_4]}$$

$$\mathscr{F}_1^2(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_2^1}} \circ F_{L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_2^1}$$

$$\mathscr{F}_1^8(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_8^7}} \circ F_{L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_8^7}$$

$$\mathscr{F}_2^3(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_3^2}} \circ F_{L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_3^2}$$

$$\mathscr{F}_2^7(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_7^6}} \circ F_{L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_7^6}$$

$$\mathscr{F}_3^4(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_4^3}} \circ F_{L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_4^3}$$

$$\mathscr{F}_3^6(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_6^5}} \circ F_{L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_6^5}$$

$$\mathscr{F}_4^{+5}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_5^3}} \circ F_{\mathrm{sym}^2 L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_5^3}$$

$$\mathscr{F}_4^{-5}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_5^3}} \circ F_{\mathrm{alt}^2 L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_5^3}$$

$$\mathscr{F}_4^{+6}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_6^4}} \circ F_{\mathrm{sym}^2 L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_6^4}$$

$$\mathscr{F}_4^{-6}(\bar{d}) := \mathrm{proj}_{\theta_{\bar{d}_6^4}} \circ F_{\mathrm{alt}^2 L_n^*}: \mathscr{O}_{\bar{d}} \to \mathscr{O}_{\bar{d}_6^4}$$

令

$$[\overrightarrow{d}_1] := [\bar{d}_2^1] = [\bar{d}_8^7],\ [\overrightarrow{d}_2] := [\bar{d}_3^2] = [\bar{d}_7^6]$$

$$[\overrightarrow{d}_3] := [\bar{d}_4^3] = [\bar{d}_6^5],\ [\overrightarrow{d}_4] := [\bar{d}_5^3] = [\bar{d}_6^4]$$

定义

$$\mathscr{F}_1([\bar{d}] := \bigoplus_{\bar{d}' \in [\bar{d}]} (\mathscr{F}_1^2(\bar{d}') \oplus \mathscr{F}_1^8(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overrightarrow{d}_1]}$$

$$\mathscr{F}_2([\bar{d}] := \bigoplus_{\bar{d}' \in [\bar{d}]} (\mathscr{F}_2^3(\bar{d}') \oplus \mathscr{F}_2^7(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overrightarrow{d}_2]}$$

$$\mathscr{F}_3([\bar{d}] := \bigoplus_{\bar{d}' \in [\bar{d}]} (\mathscr{F}_3^4(\bar{d}') \oplus \mathscr{F}_3^6(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overrightarrow{d}_3]}$$

$$\mathscr{F}_4^+([\bar{d}] := \bigoplus_{\bar{d}' \in [\bar{d}]} (\mathscr{F}_4^{+5}(\bar{d}') \oplus \mathscr{F}_4^{+6}(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overrightarrow{d}_4]}$$

$$\mathscr{F}_4^-([\bar{d}] := \bigoplus_{\bar{d}' \in [\bar{d}]} (\mathscr{F}_4^{-5}(\bar{d}') \oplus \mathscr{F}_4^{-6}(\bar{d}')): \mathscr{O}_{[\bar{d}]} \to \mathscr{O}_{[\overrightarrow{d}_4]}$$

下面考虑正合函子所诱导的阿贝尔群同态的作用.

引理 1

$$[\varepsilon_1^1(\bar{d})]([M(a_1, a_2, \cdots, a_n)]) = \sum_{\substack{m=1, \\ a_m=1}}^{n} [M(a_1, \cdots, a_{m-1}, a_m + 1, a_{m+1}, \cdots, a_n)]$$

$$[\varepsilon_1^7(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=7}}^{n}[M(a_1,\cdots,a_{m-1},a_m+1,a_{m+1},\cdots,a_n)]$$
$$[\varepsilon_2^2(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=2}}^{n}[M(a_1,\cdots,a_{m-1},a_m+1,a_{m+1},\cdots,a_n)]$$
$$[\varepsilon_2^6(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=6}}^{n}[M(a_1,\cdots,a_{m-1},a_m+1,a_{m+1},\cdots,a_n)]$$
$$[\varepsilon_3^3(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=3}}^{n}[M(a_1,\cdots,a_{m-1},a_m+1,a_{m+1},\cdots,a_n)]$$
$$[\varepsilon_3^5(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=5}}^{n}[M(a_1,\cdots,a_{m-1},a_m+1,a_{m+1},\cdots,a_n)]$$
$$[\varepsilon_4^{+3}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=3}}^{n}[M(a_1,\cdots,a_{m-1},a_m+2,a_{m+1},\cdots,a_n)] +$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(3,4)\text{or}(4,3)}}^{n}[M(a_1,\cdots,a_{i-1},a_i+1,\cdots,a_{j-1},a_j+1,a_{j+1},\cdots,a_n)]$$
$$[\varepsilon_4^{+4}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=4}}^{n}[M(a_1,\cdots,a_{m-1},a_m+2,a_{m+1},\cdots,a_n)] +$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(4,5)\text{or}(5,4)}}^{n}[M(a_1,\cdots,a_{i-1},a_i+1,a_{i+1},\cdots,a_{j-1},a_j+1,a_{j+1},\cdots,a_n)]$$
$$[\varepsilon_4^{-3}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(3,4)\text{or}(4,3)}}^{n}[M(a_1,\cdots,a_{i-1},a_i+1,a_{i+1},\cdots,a_{j-1},a_j+1,a_{j+1},\cdots,a_n)]$$
$$[\varepsilon_4^{-4}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(4,5)\text{or}(5,4)}}^{n}[M(a_1,\cdots,a_{i-1},a_i+1,a_{i+1},\cdots,a_{j-1},a_j+1,a_{j+1},\cdots,a_n)]$$
$$[\mathscr{F}_1^2(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=2}}^{n}[M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\cdots,a_n)]$$
$$[\mathscr{F}_1^8(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=8}}^{n}[M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\cdots,a_n)]$$
$$[\mathscr{F}_2^3(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=3}}^{n}[M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\cdots,a_n)]$$
$$[\mathscr{F}_2^7(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=7}}^{n}[M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\cdots,a_n)]$$
$$[\mathscr{F}_3^4(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=4}}^{n}[M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\cdots,a_n)]$$
$$[\mathscr{F}_3^6(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=6}}^{n}[M(a_1,\cdots,a_{m-1},a_m-1,a_{m+1},\cdots,a_n)]$$
$$[\mathscr{F}_4^{+5}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=5}}^{n}[M(a_1,\cdots,a_{m-1},a_m-2,\cdots,a_n)] +$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(5,4)\text{or}(4,5)}}^{n}[M(a_1,\cdots,a_{i-1},a_i-1,\cdots,a_{j-1},a_j-1,a_{j+1},\cdots,a_n)]$$
$$[\mathscr{F}_4^{-5}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(5,4)\text{or}(4,5)}}^{n}[M(a_1,\cdots,a_{i-1},a_i-1,\cdots,a_{j-1},a_j-1,a_{j+1},\cdots,a_n)]$$
$$[\mathscr{F}_4^{+6}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{m=1,\\a_m=6}}^{n}[M(a_1,\cdots,a_{m-1},a_m-2,\cdots,a_n)] +$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(5,6)\text{or}(6,5)}}^{n}[M(a_1,\cdots,a_{i-1},a_i-1,a_{i+1},\cdots,a_{j-1},a_j-1,a_{j+1},\cdots,a_n)]$$
$$[\mathscr{F}_4^{-6}(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$\sum_{\substack{1\leqslant i<j\leqslant n,\\(a_i,a_j)=(6,5)\text{or}(5,6)}}^{n}[M(a_1,\cdots,a_{i-1},a_i-1,a_{i+1},\cdots,a_{j-1},a_j-1,a_{j+1},\cdots,a_n)]$$

证明：

$$[\varepsilon_1^1(\bar{d})]([M(a_1,a_2,\cdots,a_n)]) =$$
$$[\varepsilon_1^1(\bar{d})(M(a_1,a_2,\cdots,a_n))] =$$
$$\left[\mathrm{proj}_{\theta_{\bar{d}_1^2}}\left(\sum_{i=1}^{n}M(a_1\varepsilon_1+\cdots+a_n\varepsilon_n-\rho+\varepsilon_i)\right)\right] =$$

$$\sum_{\substack{m=1,\\ a_m=1}}^{n}[M(a_1,\cdots,a_{m-1},a_m+1,a_{m+1},\cdots,a_n)]$$

其余的结论类似可证. 证毕.

定理 2

1）对任意的$[\bar{d}]\in\widetilde{D}$, $h_i(i=1,2,3,4)$在${}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$上的作用可由$\mathscr{H}_i^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}])$范畴化，即下图可换

$$\begin{array}{ccc} K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \\ {\scriptstyle [\mathscr{H}_i^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}])]}\downarrow & & \downarrow{\scriptstyle \mathrm{sgn}(c_i(\bar{d}))h_i} \\ K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \end{array}$$

2）对任意的$[\bar{d}]\in\widetilde{D}$, $e_i(i=1,2,3)$在${}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$上的作用可由$\varepsilon_i([\bar{d}])$范畴化，即下图可换

$$\begin{array}{ccc} K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \\ {\scriptstyle [\varepsilon_i([\bar{d}])]}\downarrow & & \downarrow{\scriptstyle e_i} \\ K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \end{array}$$

对任意的$[\bar{d}]\in\widetilde{D}$, e_4在${}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$的作用可由$\varepsilon_4^+([\bar{d}])$、$\varepsilon_4^-([\bar{d}])$范畴化,即下图可换

$$\begin{array}{ccc} K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \\ {\scriptstyle [\varepsilon_4^+([\bar{d}])]-[\varepsilon_4^-([\bar{d}])]}\downarrow & & \downarrow{\scriptstyle e_4} \\ K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \end{array}$$

3）对任意的$[\bar{d}]\in\widetilde{D}$,$f_i(i=1,2,3)$在${}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$上的作用可由$\mathscr{F}_i([\bar{d}])$范畴化,即下图可换

$$\begin{array}{ccc} K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \\ {\scriptstyle [\mathscr{F}_i([\bar{d}])]}\downarrow & & \downarrow{\scriptstyle f_i} \\ K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \end{array}$$

对任意的$[\bar{d}]\in\widetilde{D}$,$f_4$在${}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]}$上的作用可由$\mathscr{F}_4^+([\bar{d}])$、$\mathscr{F}_4^-([\bar{d}])$范畴化，即下图可换

$$\begin{array}{ccc} K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \\ {\scriptstyle [\mathscr{F}_4^+([\bar{d}])]-[\mathscr{F}_4^-([\bar{d}])]}\downarrow & & \downarrow{\scriptstyle f_1} \\ K(\mathscr{O}_{[\bar{d}]}) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n})_{[\bar{d}]} \end{array}$$

证明：1）任取$M(a_1,a_2,\cdots,a_n)\in B_{[\bar{d}]}$,对任意的$i=1,2,3,4$,有

$$\mathrm{sgn}(c_i(\bar{d}))h_i\circ\gamma_n([M(a_1,a_2,\cdots,a_n)])=$$

$$\mathrm{sgn}(c_i(\bar{d}))h_i(v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_n})=$$

$$\mathrm{sgn}(c_i(\bar{d}))\Big(\sum_{k=1}^{n}v_{a_1}\otimes\cdots\otimes h_iv_{a_k}\otimes\cdots\otimes v_{a_n}\Big)=$$

$$\mathrm{sgn}(c_i(\bar{d}))c_i(\bar{d})(v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_n})=$$

$$\gamma_n\circ\mathscr{H}_i^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}])([M(a_1,a_2,\cdots,a_n)])$$

故图可换. 根据定理 1 和引理 1 易证定理 2 的 2)、3)成立. 证毕.

对$1\leqslant i\leqslant 4$,$1\leqslant j\leqslant 3$,定义下面的函子

$$\varepsilon_j:=\bigoplus_{[\bar{d}]\in\widetilde{D}}\varepsilon_j([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\mathscr{F}_j:=\bigoplus_{[\bar{d}]\in\widetilde{D}}\mathscr{F}_j([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\varepsilon_4^+:=\bigoplus_{[\bar{d}]\in\widetilde{D}}\varepsilon_4^+([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\varepsilon_4^-:=\bigoplus_{[\bar{d}]\in\widetilde{D}}\varepsilon_4^-([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\mathscr{F}_4^+:=\bigoplus_{[\bar{d}]\in\widetilde{D}}\mathscr{F}_4^+([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\mathscr{F}_4^-:=\bigoplus_{[\bar{d}]\in\widetilde{D}}\mathscr{F}_4^-([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\mathscr{H}_i^+:=\bigoplus_{\substack{[\bar{d}]\in\widetilde{D}\\ c_i(\bar{d})\geqslant 0}}\mathscr{H}_i^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

$$\mathscr{H}_i^-:=\bigoplus_{\substack{[\bar{d}]\in\widetilde{D}\\ c_i(\bar{d})\geqslant 0}}\mathscr{H}_i^{\mathrm{sgn}(c_i(\bar{d}))}([\bar{d}]):\mathscr{O}^n\to\mathscr{O}^n$$

下面给出本文的主要定理.

定理 3

1）对任意的$1\leqslant i\leqslant 4$,h_i在${}^{\mathbb{Z}}(V^{\otimes n})$上的作用可由$\mathscr{H}_i^+$、$\mathscr{H}_i^-$范畴化,即下图可换

$$\begin{array}{ccc} K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \\ {\scriptstyle [\mathscr{H}_i^+]-[\mathscr{H}_i^-]}\downarrow & & \downarrow{\scriptstyle h_i} \\ K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \end{array}$$

2）对任意的$1\leqslant i\leqslant 3$,e_i在${}^{\mathbb{Z}}(V^{\otimes n})$上的作用可由$\varepsilon_i$范畴化，即下图可换

$$\begin{array}{ccc} K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \\ {\scriptstyle [\varepsilon_i]}\downarrow & & \downarrow{\scriptstyle e_i} \\ K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \end{array}$$

e_4在${}^{\mathbb{Z}}(V^{\otimes n})$上的作用可由$\varepsilon_4^+$、$\varepsilon_4^-$范畴化,即下图可换

$$\begin{array}{ccc} K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \\ {\scriptstyle [\varepsilon_4^+]-[\varepsilon_4^-]}\downarrow & & \downarrow{\scriptstyle e_4} \\ K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \end{array}$$

3）对任意的$1\leqslant i\leqslant 3$,f_i在${}^{\mathbb{Z}}(V^{\otimes n})$上的作用可由$\mathscr{F}_i$范畴化，即下图可换

$$\begin{array}{ccc} K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \\ {\scriptstyle [\mathscr{F}_i]}\big\downarrow & & \big\downarrow{\scriptstyle f_i} \\ K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \end{array}$$

f_4 在${}^{\mathbb{Z}}(V^{\otimes n})$上的作用可由 $\mathscr{F}_4^+$、$\mathscr{F}_4^-$ 范畴化，即下图可换

$$\begin{array}{ccc} K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \\ {\scriptstyle [\mathscr{F}_4^+]-[\mathscr{F}_4^-]}\big\downarrow & & \big\downarrow{\scriptstyle f_4} \\ K(\mathscr{O}^n) & \xrightarrow{\gamma_n} & {}^{\mathbb{Z}}(V^{\otimes n}) \end{array}$$

证明：根据定理 2，易证上面的图均可换．证毕．

最后范畴化定义关系．

定理 4

1) $\mathscr{H}_i^+\mathscr{H}_j^+\oplus\mathscr{H}_j^+\mathscr{H}_i^-\oplus\mathscr{H}_j^-\mathscr{H}_i^+\oplus\mathscr{H}_i^-\mathscr{H}_j^-\simeq \mathscr{H}_j^+\mathscr{H}_i^+\oplus\mathscr{H}_i^+\mathscr{H}_j^-\oplus\mathscr{H}_i^-\mathscr{H}_j^+\oplus\mathscr{H}_j^-\mathscr{H}_i^-,1\le i,j\le 4$

2) $\varepsilon_4^+\mathscr{F}_4^+\oplus\varepsilon_4^-\mathscr{F}_4^-\oplus\mathscr{F}_4^+\varepsilon_4^-\oplus\mathscr{F}_4^-\varepsilon_4^+\oplus\mathscr{H}_4^-\simeq \varepsilon_4^+\mathscr{F}_4^-\oplus\varepsilon_4^-\mathscr{F}_4^+\oplus\mathscr{F}_4^+\varepsilon_4^+\mathscr{F}_4^-\varepsilon_4^-\oplus\mathscr{H}_4^+$

$$\varepsilon_4^+\mathscr{F}_i\oplus\mathscr{F}_i\varepsilon_4^-\simeq\varepsilon_4^-\mathscr{F}_i\oplus\mathscr{F}_i\varepsilon_4^+,\ i=1,2,3$$

$$\varepsilon_i\mathscr{F}_4^+\oplus\mathscr{F}_4^-\varepsilon_i\simeq\varepsilon_i\mathscr{F}_4^-\oplus\mathscr{F}_4^+\varepsilon_i,\ i=1,2,3$$

$$\varepsilon_i\mathscr{F}_j\oplus\delta_{ij}\mathscr{H}_i^-\simeq\mathscr{F}_j\varepsilon_i\oplus\delta_{ij}\mathscr{H}_i^+,\ i,j=1,2,3$$

3) $\mathscr{H}_i^+\varepsilon_i\oplus\varepsilon_i\mathscr{H}_i^-\simeq\mathscr{H}_i^-\varepsilon_i\oplus\varepsilon_i\mathscr{H}_i^+\oplus\varepsilon_i^{\oplus2}$, $i=1,2,3$

$$\mathscr{H}_4^+\varepsilon_4^+\oplus\mathscr{H}_4^-\varepsilon_4^-\oplus\varepsilon_4^+\mathscr{H}_4^-\oplus\varepsilon_4^-\mathscr{H}_4^+\oplus(\varepsilon_4^-)^{\oplus2}\simeq \mathscr{H}_4^+\varepsilon_4^-\oplus\mathscr{H}_4^-\varepsilon_4^+\oplus\varepsilon_4^+\mathscr{H}_4^+\oplus\varepsilon_4^-\mathscr{H}_4^-\oplus(\varepsilon_4^+)^{\oplus2}$$

$$\mathscr{H}_i^+\varepsilon_j\oplus\varepsilon_j\oplus\varepsilon_j\mathscr{H}_i^-\simeq\varepsilon_j\mathscr{H}_i^+\oplus\mathscr{H}_i^-\varepsilon_j,\ (i,j)\in\{(1,2),(2,1),(2,3),(3,2),(4,2)\}$$

$$\mathscr{H}_i^+\varepsilon_j\oplus\varepsilon_j\mathscr{H}_i^-\simeq\varepsilon_j\mathscr{H}_i^+\oplus\mathscr{H}_i^-\varepsilon_j,\ (i,j)\in\{(1,3),(3,1),(4,1),(4,3)\}$$

$$\mathscr{H}_i^+\varepsilon_4^+\oplus\mathscr{H}_i^-\varepsilon_4^-\oplus\varepsilon_4^+\mathscr{H}_i^-\oplus\varepsilon_4^-\mathscr{H}_i^+\simeq \varepsilon_4^+\mathscr{H}_i^+\oplus\varepsilon_4^-\mathscr{H}_i^-\oplus\mathscr{H}_i^+\varepsilon_4^-\oplus\mathscr{H}_i^-\varepsilon_4^+,i=1,3$$

$$\mathscr{H}_2^+\varepsilon_4^+\oplus\mathscr{H}_2^-\varepsilon_4^-\oplus\varepsilon_4^+\mathscr{H}_2^-\oplus\varepsilon_4^-\mathscr{H}_2^+\oplus\varepsilon_4^+\simeq \mathscr{H}_2^+\varepsilon_4^-\oplus\mathscr{H}_2^-\varepsilon_4^+\oplus\varepsilon_4^+\mathscr{H}_2^+\oplus\varepsilon_4^-\mathscr{H}_2^-\oplus\varepsilon_4^-$$

4) $\mathscr{H}_i^+\mathscr{F}_i\oplus\mathscr{F}_i\mathscr{H}_i^-\oplus\mathscr{F}_i^{\oplus2}\simeq\mathscr{H}_i^-\mathscr{F}_i\oplus\mathscr{F}_i\mathscr{H}_i^+$, $i=1,2,3$

$$\mathscr{H}_4^+\mathscr{F}_4^+\oplus\mathscr{H}_4^-\mathscr{F}_4^-\oplus\mathscr{F}_4^+\mathscr{H}_4^-\oplus\mathscr{F}_4^-\mathscr{H}_4^+\oplus(\mathscr{F}_4^+)^{\oplus2}\simeq\mathscr{H}_4^+\mathscr{F}_4^-\oplus\mathscr{H}_4^-\mathscr{F}_4^+\oplus\mathscr{F}_4^+\mathscr{H}_4^+\oplus\mathscr{F}_4^-\mathscr{H}_4^-\oplus(\mathscr{F}_4^-)^{\oplus2}$$

$$\mathscr{H}_i^+\mathscr{F}_j\oplus\mathscr{F}_j\mathscr{H}_i^-\simeq\mathscr{H}_i^-\mathscr{F}_j\oplus\mathscr{F}_j\mathscr{H}_i^+\oplus\mathscr{F}_j,\ (i,j)\in\{(1,2),(2,1),(2,3),(3,2),(4,2)\}$$

$$\mathscr{H}_i^+\mathscr{F}_j\oplus\mathscr{F}_j\mathscr{H}_i^-\simeq\mathscr{H}_i^-\mathscr{F}_j\oplus\mathscr{F}_j\mathscr{H}_i^+,\ (i,j)\in\{(1,3),(3,1),(4,1),(4,3)\}$$

$$\mathscr{H}_i^+\mathscr{F}_4^+\oplus\mathscr{H}_i^-\mathscr{F}_4^-\oplus\mathscr{F}_4^+\mathscr{H}_i^-\oplus\mathscr{F}_4^-\mathscr{H}_i^+\simeq \mathscr{F}_4^+\mathscr{H}_i^+\oplus\mathscr{F}_4^-\mathscr{H}_i^-\oplus\mathscr{H}_i^+\mathscr{F}_4^-\oplus\mathscr{H}_i^-\mathscr{F}_4^+,i=1,3$$

$$\mathscr{H}_2^+\mathscr{F}_4^+\oplus\mathscr{H}_2^-\mathscr{F}_4^-\oplus\mathscr{F}_4^+\mathscr{H}_2^-\oplus\mathscr{F}_4^-\mathscr{H}_2^+\oplus\mathscr{F}_4^-\simeq \mathscr{H}_2^+\mathscr{F}_4^-\oplus\mathscr{H}_2^-\mathscr{F}_4^+\oplus\mathscr{F}_4^+\mathscr{H}_2^+\oplus\mathscr{F}_4^-\mathscr{H}_2^-\oplus\mathscr{F}_4^+$$

5) $\varepsilon_i\varepsilon_i\varepsilon_j\oplus\varepsilon_j\varepsilon_i\varepsilon_i\simeq(\varepsilon_i\varepsilon_j\varepsilon_i)^{\oplus2}$, $(i,j)\in\{(1,2),(2,1),(2,3),(3,2),(4,2)\}$,

$$\varepsilon_i\varepsilon_j\simeq\varepsilon_j\varepsilon_i,(i,j)\in\{(1,3),(3,1),(4,1),(4,3)\},$$

$$\varepsilon_i\varepsilon_4^+\oplus\varepsilon_4^-\varepsilon_i\simeq\varepsilon_4^+\varepsilon_i\oplus\varepsilon_i\varepsilon_4^-,i=1,3$$

$$\varepsilon_2\varepsilon_2\varepsilon_4^+\oplus(\varepsilon_2\varepsilon_4^-\varepsilon_2)^{\oplus2}\oplus\varepsilon_4^+\varepsilon_2\varepsilon_2\simeq \varepsilon_2\varepsilon_2\varepsilon_4^-\oplus(\varepsilon_2\varepsilon_4^+\varepsilon_2)^{\oplus2}\oplus\varepsilon_4^-\varepsilon_2\varepsilon_2$$

6) $\mathscr{F}_i\mathscr{F}_i\mathscr{F}_j\oplus\mathscr{F}_j\mathscr{F}_i\mathscr{F}_i\simeq(\mathscr{F}_i\mathscr{F}_j\mathscr{F}_i)^{\oplus2}$, $(i,j)\in\{(1,2),(2,1),(2,3),(3,2),(4,2)\}$

$$\mathscr{F}_i\mathscr{F}_j\simeq\mathscr{F}_j\mathscr{F}_i,(i,j)\in\{(1,3),(3,1),(4,1),(4,3)\}$$

$$\mathscr{F}_i\mathscr{F}_4^+\oplus\mathscr{F}_4^-\mathscr{F}_i\simeq\mathscr{F}_4^+\mathscr{F}_i\oplus\mathscr{F}_i\mathscr{F}_4^-,i=1,3$$

$$\mathscr{F}_2\mathscr{F}_2\mathscr{F}_4^+\oplus(\mathscr{F}_2\mathscr{F}_4^-\mathscr{F}_2)^{\oplus2}\oplus\mathscr{F}_4^+\mathscr{F}_2\mathscr{F}_2\simeq \mathscr{F}_2\mathscr{F}_2\mathscr{F}_4^-\oplus(\mathscr{F}_2\mathscr{F}_4^+\mathscr{F}_2)^{\oplus2}\oplus\mathscr{F}_4^-\mathscr{F}_2\mathscr{F}_2$$

证明：由于证明方法类似，只证明

$$\mathscr{H}_i^+\varepsilon_4^+\oplus\mathscr{H}_i^-\varepsilon_4^-\oplus\varepsilon_4^+\mathscr{H}_i^-\oplus\varepsilon_4^-\mathscr{H}_i^+\simeq \varepsilon_4^+\mathscr{H}_i^+\oplus\varepsilon_4^-\mathscr{H}_i^-\oplus\mathscr{H}_i^+\varepsilon_4^-\oplus\mathscr{H}_i^-\varepsilon_4^+,i=1,3$$

因为 $a_{i4}=0$，根据定义 1 有 $h_ie_4=e_4h_i$．对应地有

$$([\mathscr{H}_i^+]-[\mathscr{H}_i^-])([\varepsilon_4^+]-[\varepsilon_4^-])=([\varepsilon_4^+]-[\varepsilon_4^-])([\mathscr{H}_i^+]-[\mathscr{H}_i^-])$$

整理得

$$[\mathscr{H}_i^+][\varepsilon_4^+]+[\mathscr{H}_i^-][\varepsilon_4^-]+[\varepsilon_4^+][\mathscr{H}_i^-]+[\varepsilon_4^-][\mathscr{H}_i^+]=[\varepsilon_4^+][\mathscr{H}_i^+]+[\varepsilon_4^-][\mathscr{H}_i^-]+[\mathscr{H}_i^+][\varepsilon_4^-]+[\mathscr{H}_i^-][\varepsilon_4^-]$$

根据命题 1 知结论成立．证毕．

3 结论

1) 定义了函子 $\mathscr{H}_i^+$、$\mathscr{H}_i^-$ $(1\le i\le 4)$，范畴化了 h_i 在${}^{\mathbb{Z}}(V^{\oplus n})$上的作用．

2) 定义了函子 ε_i $(1\le i\le 3)$，范畴化了 e_i 在${}^{\mathbb{Z}}(V^{\oplus n})$上的作用；定义了函子 ε_4^+、ε_4^-，范畴化了 e_4 在${}^{\mathbb{Z}}(V^{\oplus n})$上的作用．

3) 定义了函子 $\mathscr{F}_i$ $(1\le i\le 3)$，范畴化了 f_i 在${}^{\mathbb{Z}}(V^{\oplus n})$上的作用；定义了函子 $\mathscr{F}_4^+$、$\mathscr{F}_4^-$，范畴化了 f_4 在${}^{\mathbb{Z}}(V^{\oplus n})$上的作用．

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